UT-04-01
hep-th/0401184

Five-dimensional Chern–Simons terms and Nekrasov's instanton counting

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abstract

We extend the graviphoton-corrected prepotential of 5D pure $U(N)$ super Yang-Mills, which was originally proposed by Nekrasov, by incorporating the effect of the 5D Chern-Simons term. This extension allows us to reproduce by a gauge theory calculation the partition functions of corresponding topological A-model on local toric Calabi-Yau manifolds X_N^m for all $m = 0, 1, \ldots, N$. The original proposal corresponds to the case $m = 0$.

1 Introduction

The determination of low-energy prepotential of four-dimensional (4D) $\mathcal{N} = 2$ super Yang-Mills theory initiated by Seiberg and Witten marked a significant step toward our understanding of non-perturbative dynamics of gauge theory. The results were originally derived by exploiting holomorphy and electromagnetic duality. Recently, several authors[1, 2, 3] noticed that the action functional of the constrained instanton calculus for $\mathcal{N} = 2$ super Yang-Mills has the form of cohomological field theory, to which powerful localization technique is applicable. Based on this observation, Nekrasov and collaborators determined all the instanton corrections and showed that they agree with the result obtained from the Seiberg-Witten curve[4, 5]. He also proposed a simple extension of his formula for the exact graviphoton-corrected prepotential for the five-dimensional (5D) pure $U(N)$ super Yang-Mills theory.

We have also seen a great amount of development on the side of topological strings in the last two years. The advent of the so-called topological vertex enabled us to calculate all-genus topological A-model amplitude for any local toric Calabi-Yau using certain kind of Feynman-like rules. According to [6, 7], the topological amplitude should be equal to the graviphoton-corrected prepotential of the physical theory obtained by compactifying the type IIA string (or more precisely M-theory) on the same Calabi-Yau. Indeed, Iqbal and Kashani-Poor found [8, 9], by using some physically-motivated simplifying assumptions, that for certain local toric Calabi-Yau X_N^0 the resulting all-genus amplitude exactly agreed with the 5D prepotential proposed by Nekrasov. Later, Eguchi and Kanno [10, 11] showed in a mathematically rigorous way the equivalence of the two expressions. They also extended the results to cases with various matter contents.

However, certain mysteries remain. There are $N + 1$ local toric Calabi-Yau manifolds X_N^m ($m = 0, 1, \ldots, N$) which can be used to geometrically engineer the same 4D $\mathcal{N} = 2$ $U(N)$ super Yang-Mills theories. As was shown in [9], only for $m = 0$ the topological amplitude agreed with the 5D prepotential proposed by Nekrasov. What kind of physical mechanism is operating behind the discrepancy for nonzero m? We solve this puzzle in this short note. Namely, we exactly reproduce the all-genus closed string amplitudes for X_N^m with m nonzero in a gauge theory calculation *à la* Nekrasov. It is done by properly taking into account the effects of 5D Chern-Simons terms.

The rest of the paper is organized as follows. In section 2, we very briefly review the instanton counting by Nekrasov. In section 3, we review the relation between the 5D Chern-Simons term and the triple intersection of the Calabi-Yau. After these preparations, we analyze the effect of 5D Chern-Simons terms to the instanton counting in section 4. Finally, in section 5, we conclude by discussing some of the future directions. In the appendix we discuss the relation between the partition functions on the Ω background and the graviphoton-corrected prepotential.

2 Brief Review of Nekrasov's Instanton Counting

Let us recapitulate briefly the method of computation of the graviphoton-modified prepotential, as presented by Nekrasov. We consider 5D pure Yang-Mills theory with eight supercharges. We

put it on the so-called Ω background with the metric

$$ds^2 = (dx^5)^2 + (dx^\mu + A^\mu dx^5)^2, \tag{1}$$

where $\Omega_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ is constant anti-self-dual(ASD), and the circumference of the fifth direction is β. We denote the eigenvalues of $\Omega_{\mu\nu}$ by $\pm\hbar$.

Firstly, we encode the vacuum expectation value of the adjoint scalar as the Wilson line $\exp(\beta\,\mathrm{diag}(a_1,\ldots,a_N))$ around the fifth direction. This is possible because 4D scalar field in the vector multiplet consists of one real 5D scalar and the Wilson line around the circle. From holomorphy, we have only to calculate the prepotential with Wilson line turned on, with no vacuum expectation value for the 5D scalar.

Secondly, we note that the partition function Z of supersymmetric theories on the Ω background and the graviphoton-corrected prepotential $\mathscr{F}_{\text{grav}}$ of the same theory generically satisfy the relation

$$\mathscr{F}_{\text{grav}} = \hbar^2 \log Z. \tag{2}$$

This relation was shown originally in [4]. Another derivation using the Hilbert scheme of points is presented in the appendix. Z can be schematically expressed as

$$Z = \mathrm{tr}(-)^F e^{-\beta H} e^{\beta\Omega_{\mu\nu}J^{\mu\nu}} e^{\beta a_i J^i}, \tag{3}$$

where H is the Hamiltonian of the field theory, $J^{\mu\nu}$ the generators of $SO(4)$, and J^i are the generators of global $U(N)$ gauge rotations.

Thirdly, it is argued that the calculation of the partition function localizes onto the moduli space of instantons. Thus Z is given by

$$Z = \sum_k q^k\, \mathrm{tr}(-)^F e^{-\beta H_k} e^{-\beta\Omega_{\mu\nu}J^{\mu\nu}} e^{-\beta a_i J^i} \tag{4}$$

where $q = e^{2\pi i\beta\tau}$ counts the instanton number. Now H_k is the Hamiltonian of the supersymmetric quantum mechanics on the "framed" k-instanton moduli space $\mathscr{M}_{N,k}$ of dimension $4Nk$. From the index theorem, there are $4Nk$ real fermionic adjoint zero-modes around the k-instanton configuration. Hence, the Hilbert space of the quantum mechanical system is the space of sections of the spin bundle of the instanton moduli. Thus, the trace in equation (4) can be identified with the equivariant index of the k-instanton moduli. We focus on the Cartan subgroup $U(1)^{2+N} \subset SO(4) \times U(N)$ of the global symmetries.

We use the following fixed point theorem to calculate the equivariant index:

Theorem Let M be a spin manifold acted by an abelian group G. Take an element a from the Lie algebra of G and let $g = e^{\beta a}$. Assume the fixed points of G are isolated. Then

$$\mathrm{tr}(-)^F g = \sum_{\substack{p:\text{fixed}\\ \text{points of } G}} \prod_{\substack{w:\text{eigenvalues}\\ \text{of } a \text{ on } TM_p}} \frac{1}{2\sinh\frac{\beta}{2}w} \tag{5}$$

where the left hand side is traced over zero-modes of the Dirac operators on M.

This theorem reduces the calculation of Z to the study of fixed points in the moduli space. The fixed points are at the small instanton singularities in $\mathscr{M}_{N,k}$. Hence we need to use the moduli of noncommutative instantons $\widetilde{\mathscr{M}_{N,k}}$ as the ultraviolet regularization. The result turns out to be independent of the noncommutativity parameter. Fixed points in $\widetilde{\mathscr{M}_{N,k}}$ was studied by Nekrasov [4] and Nakajima and Yoshioka[12]. They are labeled by N-tuples of Young tableaux $(Y_1, \ldots, Y_N)$. We denote the number of boxes of the i-th row of the tableau Y by y_i. The action of g on the tangent space at the fixed points can be studied straightforwardly, and gives the celebrated formula of Nekrasov

$$\exp(\hbar^{-2}\mathscr{F}_{\text{grav}}) = \sum_{Y_1,\ldots,Y_N} \left(\frac{q}{4^N}\right)^{\sum \ell_{Y_i}} \prod_{l,n=1}^{N} \prod_{i,j=1}^{\infty} \frac{\sinh\frac{\beta}{2}(a_{ln} + \hbar(y_{l,j} - y_{n,i} + j - i))}{\sinh\frac{\beta}{2}(a_{ln} + \hbar(y_{l,j} - y_{n,i}))} \tag{6}$$

where ℓ_Y is the number of boxes in Y, $\hbar$ is the magnitude of $\Omega_{\mu\nu}$, and a_{ij} denotes $a_i - a_j$. Note that the result is independent under the constant shift $a_i \to a_i + c$.

To determine the g action on the tangent space, we need to use the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction of instantons. Let $\mathbb{X}$ denote the vector space

$$\mathbb{X} = (V^* \otimes V) \oplus (V^* \otimes V) \oplus (W^* \otimes V) \oplus (V^* \otimes W) \tag{7}$$

where $V = \mathbb{C}^k$ and $W = \mathbb{C}^N$. Define a $U(k)$ action on $\mathbb{X}$ by letting V and W transform in the fundamental and trivial representations respectively. $\widetilde{\mathscr{M}_{N,k}}$ is the hyperkähler quotient of $\mathbb{X}$ by the $U(k)$ action. Hence, a fixed point $p = gp$ in $\widetilde{\mathscr{M}_{N,k}}$ corresponds to an ADHM datum $x_p \in X$ fixed up to the $U(k)$ action:

$$gx_p = \phi_p(g)x_p \tag{8}$$

where $g \in U(1)^{2+N}$ and $\phi_p(g) \in U(k)$. An essential part of the calculation leading to the formula (6) is the property that at the fixed point labeled by $(Y_1, Y_2, \ldots, Y_N)$, $\phi(g)$ has k eigenvalues given by

$$\exp(-\beta(a_l + \hbar(i - j))) \quad \text{for each box } (i, j) \in Y_l \tag{9}$$

where l runs from $1, \ldots, N$.

3 Triple intersection and the Chern-Simons terms

In this section we recapitulate how the 5D theories obtained by M-theory compactification on X_N^m differ from each other. Firstly let us recall the generic structure of 5D supersymmetric $U(1)^n$ gauge theory with eight supercharges[13]. Classically, the theory is specified by the prepotential $\mathscr{F} = c_{ijk}a_i a_j a_k + \tau_{ij}a_i a_j$ of degree up to three. It is because the third derivative of $\mathscr{F}$ gives the coefficient of the Chern-Simons term

$$\int c_{ijk}A_i \wedge F_j \wedge F_k. \tag{10}$$

This will not be gauge-invariant unless the third derivative is constant.

Secondly let us see how the coefficients c_{ijk} is determined from the geometric data, when the theory is realized by a M-theory compactification. In this setup, 5D vector fields A_i come from the three-form field $C^{(3)}$ of the eleven-dimensional supergravity reduced along two-cycles C_i in the Calabi-Yau, $A_i = \int_{C_i} C^{(3)}$. The 5D Chern-Simons term (10) comes directly from the eleven dimensional Chern-Simons coupling

$$\int C^{(3)} \wedge (dC)^{(4)} \wedge (dC)^{(4)}. \tag{11}$$

Thus we see that the coefficient c_{ijk} is precisely the triple intersection of (the Poincaré duals of) two-cycles C_i.

When the Calabi-Yau space develops an ADE singularity through the collapse of two cycles, there appears enhanced non-abelian gauge symmetry corresponding to the ADE type of the singularity. In such cases, the Chern-Simons coupling (10) should be likewise enhanced to the non-abelian version $CS(A, F)$ which is defined through the descent equation

$$dCS(A, F) = \mathrm{tr}(F \wedge F \wedge F) \tag{12}$$

where F is the non-abelian field strength. Moreover, Intriligator *et.al.* [14] studied the geometry of general Calabi-Yau manifolds which give rise to 5D $U(N)$ theories and showed that the triple intersection is determined up to the coefficient of this non-abelian Chern-Simons term.

Iqbal and Kashani-Poor studied M-theory compactification on local toric Calabi-Yau manifolds X^m_N [9]. We collect here relevant facts on those manifolds. See [9, 14] for more detailed accounts. X^m_N is a fibration of A_{N-1} singularity over the base $\mathbb{CP}^1$. It contains a sequence of compact divisors

$$S_1 = \mathbb{F}_{m+2-N}, \quad S_2 = \mathbb{F}_{m+4-N}, \quad \ldots, \quad S_{N-1} = \mathbb{F}_{m+N-2} \tag{13}$$

Here $\mathbb{F}_n$ denotes a Hirzebruch surface. The Hirzebruch surface $\mathbb{F}_m$ is a $\mathbb{CP}^1$ fibration over the $\mathbb{CP}^1$ with the intersection pairing

$$B \cdot B = -m, \qquad B \cdot F = 1, \qquad F \cdot F = 0 \tag{14}$$

where B and F denote the base and the fiber, respectively. An A_{N-1} singularity contains at the tip $N-1$ $\mathbb{CP}^1$'s $C_1, \ldots, C_{N-1}$ with intersection pairing $C_i \cdot C_{i+1} = 1$. The divisor S_i of X^m_N is the fibration of C_i over the base $\mathbb{CP}^1$. The classical prepotential, or equivalently the triple intersection is given by the formula

$$\mathscr{F} = \frac{1}{2} \sum_{i,j} |a_i - a_j|^3 + m \sum a_i^3 \tag{15}$$

where we defined the basis $a_1, \ldots, a_N$ of special coordinates by

$$\mathscr{F} = \sum_{i,j,k} (a_{i+1} + \cdots + a_N)(a_{j+1} + \cdots + a_N)(a_{k+1} + \cdots + a_N)(S_i \cdot S_j \cdot S_k). \tag{16}$$

This is a natural choice since S_i corresponds to the simple root $a_i - a_{i+1}$. From these expressions we see that the label m of X^m_N is exactly proportional to the magnitude of the 5D Chern-Simons term.

4 Effect of the Five-dimensional Chern-Simons term

We saw in the last section that the M-theory compactifications on manifolds X_N^m have different coefficients for the non-abelian 5D Chern-Simons terms for different m. Let us next see how the presence of 5D Chern-Simons terms changes the derivation of the graviphoton-corrected prepotential reviewed in section 2.

4.1 Calculation *à la* Nekrasov

Let us calculate the graviphoton-corrected prepotential of 5D $U(N)$ super Yang-Mills with m units of non-abelian Chern-Simons term. Firstly we put the theory on the Ω background and encode the moduli of the theory by introducing Wilson lines along the fifth direction. The calculation is then localized to that of supersymmetric quantum mechanics on the ASD instanton moduli space. The Lagrangian of the quantum mechanical system is obtained by substituting the gauge field in the 5D action by the corresponding anti-self-dual configurations specified by the trajectory in the moduli space. The Yang-Mills action gives the kinetic term for the point particle moving on the ASD moduli, and the Chern-Simons term gives a phase depending on the trajectory:

$$e^{im\int CS(A,F)} = e^{im\int dx^i \mathscr{A}_i} \tag{17}$$

where $x^\mu(t)$ is the trajectory in the moduli space and the point particle is now coupled with an external vector potential $\mathscr{A}$. Therefore, the exact partition function of the theory put on the Ω background is

$$Z_{\text{gauge}} = \sum_k q^k \operatorname{tr}(-)^F e^{-\beta H_k} e^{-\beta\Omega_{\mu\nu}J^{\mu\nu}} e^{-\beta a_i J^i}. \tag{18}$$

Now H_k is the Hamiltonian of the supersymmetric quantum mechanics on $\mathscr{M}_{N,k}$ coupled to the external gauge potential $\mathscr{A}$. The Hilbert space of the system is the space of sections of $S \otimes L$, where S is the spin bundle of $\mathscr{M}_{N,k}$ and L is the line bundle determined by $\mathscr{A}$. Now we use the extended version of the fixed point theorem(see *e.g.* [15]):

Theorem Let $E \to M$ be a vector bundle over a spin manifold with an action by an abelian group G. Let a an element of the Lie algebra of G and take $g = e^{\beta a}$. Assume the fixed points of G are isolated. Then

$$\operatorname{tr}(-)^F g = \sum_{p\text{:fixed points of } G} (\operatorname{tr} g|_{E_p}) \prod_{\substack{w\text{:eigenvalues}\\ \text{of } a \text{ on } TM_p}} \frac{1}{2\sinh\frac{\beta}{2}w} \tag{19}$$

where the trace in the left hand side is taken over the zero-modes of the Dirac operator on the spin bundle tensored by E.

In view of the theorem, the study of g action on $L|_p$ suffices to determine Z, since we know the placement of fixed points and g action on the tangent spaces already. The line bundle L has long been known to physicists. It is the determinant line bundle $\operatorname{Det} \not{D}$. The determinant line bundle is defined on the space of connections $\mathscr{A}/\mathscr{G}$ and the fiber at a configuration A is defined by $(\det \operatorname{Ker} \not{D}_A)^* \otimes \det \operatorname{Ker} \not{D}_A^\dagger$ where D_A is the chiral Dirac operator coupled to the connection A

in the fundamental representation. When the base space is restricted to the ASD moduli space, it can be simplified to $(\det \mathrm{Ker}\, \not{D}_A)^*$ because we know that there are no wrong chirality zero-modes. Close relation between the determinant line bundle and the Chern-Simons terms has long been known since the work of Alvarez-Gaumé and Ginsparg [16] on the geometric interpretation of non-abelian anomalies.

In reality we need to blow up the small instanton singularity in the ASD moduli using spacetime non-commutativity. Hence we need the non-commutative extension of all these relations among anomaly, index theorem and the Chern-Simons terms. Fortunately every detail we need has already been worked out by various groups following the seminal work of Seiberg and Witten on noncommutativity. We refer the reader the works [17, 18] for noncommutative extension of the relation of non-abelian anomalies, Chern-Simons terms and the index theorem in six dimesions, and the work [19] for the study of the Dirac zero-modes in the non-commutative instanton background. We know from these works that there is no essential difference between commutative and non-commutative spacetime with regard to the relation of anomaly and the Chern-Simons terms.

Now that we have clear understanding on the nature and the structure of the line bundle L, we can complete the calculation. The fiber at p is the highest exterior power of the kernel of the Dirac operator. Thus, to determine the weight w, we have to determine the action of g on the Dirac zero-modes. As reviewed in section 2, the ADHM datum x_p itself is not invariant under the action of g, it maps x_p to a datum equivalent under $U(k)$ transformation ϕ:

$$g x_p = \phi(g) x_p. \tag{20}$$

Furthermore, we know from the analysis in [19] that k Dirac zero-modes in the fundamental representation of $U(N)$ transforms as a fundamental representation of $U(k)$. One way of seeing this is to note that when one reconstructs the ADHM datum from an ASD connection, the k-dimensional vector space V in equation (7) is none other than the space of zero-modes of the Dirac equation. These arguments show that the action of g on the zero-modes can be traded by the action of $\phi(g)$. Hence $g|_{L_p}$ can be readily computed to give

$$g|_{L_p} = \exp\left(-\beta \sum_k \sum_{(i,j)\in Y_k} (a_k + \epsilon(i-j))\right) = \exp\left(-\beta \sum_k (\ell_{Y_k} a_k + \hbar \kappa_{Y_k})\right) \tag{21}$$

where $(Y_1, \ldots, Y_N)$ labels the fixed points and we defined

$$\kappa_Y \equiv \sum_{(i,j)\in Y} (i-j) = \sum_i y_i (y_i + 1 - 2i). \tag{22}$$

Combining all these, we get the partition function for the 5D theory with non-abelian Chern-Simons term:

$$Z_{\text{gauge}} = \sum_{Y_1,\ldots,Y_N} \left(\frac{q}{4^N}\right)^{\sum_i \ell_{Y_i}} e^{-m\beta \sum (\ell_{Y_i} a_i + \hbar \kappa_{Y_i})} \prod_{l,n=1}^{N} \prod_{i,j=1}^{\infty} \frac{\sinh \frac{\beta}{2}(a_{ln} + \hbar(y_{l,j} - y_{n,i} + j - i))}{\sinh \frac{\beta}{2}(a_{ln} + \hbar(y_{l,j} - y_{n,i}))}. \tag{23}$$

We defined $a_{ij} = a_i - a_j$ for brevity. Note that the combined transformation

$$a_i \to a_i + c, \qquad 2\pi i\tau \to 2\pi i\tau - mc \tag{24}$$

does not change the result as it should be.

4.2 Comparison with the topological A-model amplitudes

Let us compare what we have obtained *à la* Nekrasov against the topological A-model amplitudes for local toric Calabi-Yau manifolds X_N^m. Combining the equations (67,68,69,78) in the article by Iqbal and Kashani-Poor[9] and changing their notation to ours, the amplitude is

$$
\begin{aligned}
Z_{\text{top.}} = \sum_{Y_1,\ldots,Y_N} & 2^{-2N\sum \ell_{Y_i}} (-)^{(N+m)\sum \ell_{Y_i}} q^{\frac{1}{2}\sum_{i=1}^{N}(N+m-2i)\kappa_i} Q_B^{\sum \ell_i} \times \\
& \prod_{i=1}^{\lfloor \frac{N+m-1}{2} \rfloor} Q_i^{(N+m-2i)(\ell_1+\cdots+\ell_i)} \prod_{i=\lfloor \frac{N+m+1}{2} \rfloor}^{N-1} Q_i^{(2i-m-N)(\ell_{i+1}+\cdots+\ell_N)} \prod_{i=1}^{N-1} Q_{b_i}^{-(N-i)(\ell_1+\cdots+\ell_i)-i(\ell_{i+1}+\cdots+\ell_N)} \\
& \times e^{-\frac{1}{2}\beta\hbar\sum_{i=1}^{N}(N-2i)\kappa_i} \prod_{l,n=1}^{N} \prod_{i,j=1}^{\infty} \frac{\sinh\frac{\beta}{2}(a_{ln}+\hbar(y_{l,j}-y_{n,i}+j-i))}{\sinh\frac{\beta}{2}(a_{ln}+\hbar(y_{l,j}-y_{n,i}))}
\end{aligned}
\tag{25}
$$

where Q_B and $Q_i = e^{-\beta(a_i-a_{i+1})}$ are respectively the exponential of the Kähler parameters of the base divisor and the divisors S_i. Define a_N by

$$
e^{-\beta a_N} = -\left(Q_B \prod_{i=1}^{\lfloor \frac{N+m-1}{2} \rfloor} Q_i^{-i} \prod_{i=\lfloor \frac{N+m+1}{2} \rfloor}^{N-1} Q_i^{-(N-i+m)} \right)^{\frac{1}{m}}. \tag{26}
$$

Then, after reshuffling the various factors with some effort, one finds that

$$
Z_{\text{top.}} = \sum_{Y_1,\ldots,Y_N} \frac{1}{(-4)^{N\sum \ell_{Y_i}}} e^{-m\beta\sum(\ell_{Y_i}a_i+\hbar\kappa_{Y_i})} \prod_{l,n=1}^{N} \prod_{i,j=1}^{\infty} \frac{\sinh\frac{\beta}{2}(a_{ln}+\hbar(y_{l,j}-y_{n,i}+j-i))}{\sinh\frac{\beta}{2}(a_{ln}+\hbar(y_{l,j}-y_{n,i}))}. \tag{27}
$$

Using the global symmetry (24), we see that this precisely agrees with the calculation of the gauge theory side *à la* Nekrasov, equation (23).

5 Conclusion

In this short note, we extended the 5D graviphoton-corrected prepotential proposed by Nekrasov to include the effect of 5D non-abelian Chern-Simons term. We saw that the introduction of 5D Chern-Simons terms results in twisting of the spin bundle on the instanton moduli by the determinant line bundle. We obtained using the fixed point theorem the partition function of the 5D $U(N)$ super Yang-Mills theory with 5D Chern-Simons term on the Ω background. Moreover we saw that the result precisely reproduced the topological string amplitude originally obtained by Iqbal and Kashani-Poor and mathematically proved by Eguchi and Kanno. This agreement is as it should be, because the partition function of the topological A-model on a Calabi-Yau and that of M-theory compactified on the same Calabi-Yau times the Ω background should be equal.

It will be worthwhile to generalize the calculation *à la* Nekrasov to general 5D theories obtained from M-theory compactification on Calabi-Yau manifolds. It will be extremely interesting

if we can generically prove the equality of the gauge theory partition function with the topological string partition function by extending Nekrasov's instanton counting. As the topological string amplitudes can be calculated from the topological vertex, it will be tempting to suggest the existence of some kind of 'gauge vertex' which upon gluing yields general gauge theory partition functions.

Acknowledgment The author thanks to K. Sakai, R. Nobuyama, Prof. T. Eguchi and Prof. H. Kanno for very stimulating discussion on the subject.

Appendix: Derivation of the relation (2)

In this appendix we give yet another derivation of the relation (2). Firstly recall the argument presented by Gopakumar and Vafa. They showed in [20, 21] that an BPS multiplet with the left spin content

$$I_r = \left((\frac{1}{2}) \oplus 2(0) \right)^{r+1} \tag{28}$$

and with central charge a contributes to the graviphoton-corrected prepotential by an amount

$$\hbar^{-2} \mathscr{F}_r(a) = \sum_{k>0} \frac{1}{k} (2 \sinh \frac{k\hbar}{2})^{2r-2} e^{-ka} \tag{29}$$

where $\hbar$ is the magnitude of the field strength of the graviphoton. This equation can be proved using the Fock-Schwinger proper time method. Another convenient basis of the left spin content is

$$C_j = \left((\frac{1}{2}) \oplus 2(0) \right) \otimes (\text{a state with } J_L^3 = j) \tag{30}$$

In this basis, the contribution to the prepotential becomes

$$\begin{aligned} \hbar^{-2} \mathscr{F}_j(a) &= \sum_{k>0} \frac{1}{k} \frac{1}{(2 \sinh k\hbar/2)^2} e^{-k(a+2j\hbar)} \qquad (31) \\ &= \sum_{n>0} \log \left(1 - e^{-(a+2j\hbar+n\hbar)} \right). \qquad (32) \end{aligned}$$

The prepotential is a quantity protected by supersymmetry and receives contributions only from states annihilated by half of the supersymmetry, *i.e.* BPS states. Thus the exact prepotential of the low energy theory is given by

$$\hbar^{-2} \mathscr{F} = \sum_{i,r} N_{i,r} \sum_{k>0} \frac{1}{k} (2 \sinh \frac{k\hbar}{2})^{2r-2} e^{-ka_i} \tag{33}$$

where $N_{i,r}$ is the number of multiplets with central charge a_i and spin content I_r. $N_{i,r}$ is called the Gopakumar-Vafa invariants of the theory.

Next consider the partition function of the theory on the Ω background. Let us canonically quantize the theory, considering the fifth direction dx^5 as the time direction. Then, in the Hamiltonian formalism, the partition function can be schematically written as

$$Z = \mathrm{tr}(-)^F e^{-\beta H} \exp(i\beta\Omega_{\mu\nu} J^{\mu\nu}). \tag{34}$$

where H is the total Hamiltonian of the field theory. This is none other than the equivariant index of the system. $\exp(i\beta\Omega_{\mu\nu} J^{\mu\nu})$ commutes with half of the supersymmetry when the curvature $\Omega_{\mu\nu}$ is self-dual. Thus, the partition function Z receives contributions only from the states annihilated by those supersymmetry. These states are precisely what contributed to the prepotential of the theory put on the graviphoton background. These consideration reveals us that the partition function can be written as an infinite product of the form

$$Z = \prod_{r,i} Z_r(a_i)^{N_{i,r}} \tag{35}$$

where $N_{i,r}$ is the same Gopakumar-Vafa invariants we discussed above. Hence, we need only to show

$$Z_r(a) = \exp(\hbar^{-2}\mathscr{F}_r(a)) \tag{36}$$

in order to show the relation (2).

Let us show the relation (36) for the case $r = 0$. The extension to other cases should be immediate. Hence we are going to calculate the partition function of a free hypermultiplet on the Ω background. In a first-quantized framework, the system is thought of as a collection of particles and anti-particles. The calculation of the partition function is localized by the supersymmetry to the configuration space of BPS states. A BPS configuration is a collection of particles only, since an anti-particle respects the other half of the supersymmetry and particle-antiparticle pair breaks all of the supersymmetry. As the particles are indistinguishable from each other, the configuration space of k particles is

$$S^k\mathbb{C}^2 \equiv (\mathbb{C}^2)^k/\mathfrak{S}_k. \tag{37}$$

Hence, the partition function should be

$$Z = \sum e^{-ka}\,\mathrm{Ind}_g\, S^k\mathbb{C}^2. \tag{38}$$

where $\mathrm{Ind}_g\, M$ denotes the equivariant index, i.e. the trace of g over the space of harmonic spinors of M. However, the space $S^k\mathbb{C}^2$ is singular and reliable calculation of the equivariant index is difficult. It is known that there is a good resolution $(\mathbb{C}^2)^{[k]}$ of the singularities of $S^k\mathbb{C}^2$, called the Hilbert scheme of k-points on $\mathbb{C}^2$. Furthermore, it is known to be identical to $\widetilde{M_{1,k}}$, the moduli of non-commutative $U(1)$ instantons. Hence, the fixed points in $(\mathbb{C}^2)^{[k]}$ is labeled by a Young tableau Y and the result is

$$Z = \sum_Y e^{-\ell_Y a} \prod_{s\in Y} \left(\frac{1}{\sinh \frac{\beta}{2}\hbar(l(s) + a(s) + 1)} \right)^2 \tag{39}$$

where we defined for a Young tableau its arm length and leg length by

$$a_Y(s) = y_i - j, \qquad\qquad l_Y(s) = y_j^D - i. \tag{40}$$

for a box $s = (i, j) \in Y$. Y^D denotes the transpose of the Young tableau Y. We refer the reader to the lecture notes by H. Nakajima[22] for a detailed derivation. The expression (39) can be simplified using the free fermion technique [11] to give

$$Z = \prod_{n\geq 1}\left(\frac{1}{1 - e^{-a}e^{-\hbar n}}\right)^n = \exp(\hbar^{-2}\mathscr{F}_{r=0}(a)). \tag{41}$$

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